arXiv:1001.1651v1 [math.OA] 11 Jan 2010

COMPARISON OF TOPOLOGIES ON $*$-ALGEBRAS OF LOCALLY MEASURABLE OPERATORS

V. I. CHILIN AND M. A. MURATOV

Abstract. We consider the locally measure topology $t(\mathcal{M})$ on the $*$-algebra $LS(\mathcal{M})$ of all locally measurable operators affiliated with a von Neumann algebra $\mathcal{M}$. We prove that $t(\mathcal{M})$ coincides with the (o)-topology on $LS_h(\mathcal{M}) = \{T \in LS(\mathcal{M}) : T^* = T\}$ if and only if the algebra $\mathcal{M}$ is σ-finite and a finite algebra. We study relationships between the topology $t(\mathcal{M})$ and various topologies generated by faithful normal semifinite traces on $\mathcal{M}$.

Introduction

The development of integration theory for a faithful normal semifinite trace τ defined on a von Neumann algebra $\mathcal{M}$ has led to a need to consider the $*$-algebra $S(\mathcal{M},\tau)$ of all τ-measurable operators affiliated with $\mathcal{M}$, see, e.g., [1]. This algebra is a solid $*$-subalgebra of the $*$-algebra $S(\mathcal{M})$ of all measurable operators affiliated with $\mathcal{M}$. The $*$-algebra $S(\mathcal{M})$ was introduced by I. Segal [2] to describe a "noncommutative version" of the $*$-algebra of measurable complex-valued functions. If $\mathcal{M}$ is a commutative von Neumann algebra, then $\mathcal{M}$ can be identified with the $*$-algebra $L_\infty(\Omega,\Sigma,\mu)$ of all essentially bounded measurable complex-valued functions defined on a measure space (Ω,Σ,μ) with a measure μ having the direct sum property. In this case, the $*$-algebra $S(\mathcal{M})$ is identified with the $*$-algebra $L_0(\Omega,\Sigma,\mu)$ of all measurable complex-valued functions defined on (Ω,Σ,μ) [2].

The $*$-algebras $S(\mathcal{M},\tau)$ and $S(\mathcal{M})$ are substantive examples of EW^*-algebras E of closed linear operators, affiliated with the von Neumann algebra $\mathcal{M}$, which act on the same Hilbert space $\mathcal{H}$ as $\mathcal{M}$ and have the bounded part $E_b = E \cap \mathcal{B}(\mathcal{H})$ coinciding with $\mathcal{M}$ [3], where $\mathcal{B}(\mathcal{H})$ is the $*$-algebra of all bounded linear operators on $\mathcal{H}$. A natural desire of obtaining a maximal EW^*-algebra E with $E_b = \mathcal{M}$ has led to a construction of the $*$-algebra $LS(\mathcal{M})$ of all locally measurable operators affiliated with the von Neumann algebra $\mathcal{M}$, see, for example, [4]. It was shown in [5] that any EW^*-algebra E satisfying $E_b = \mathcal{M}$ is a solid $*$-subalgebra of $LS(\mathcal{M})$.

In the case where there exists a faithful normal finite trace τ on $\mathcal{M}$, all three $*$-algebras $LS(\mathcal{M})$, $S(\mathcal{M})$, and $S(\mathcal{M},\tau)$ coincide [6, § 2.6], and a

2000 *Mathematics Subject Classification.* 46L50, 47D25, 47D40, 06F25, 06F30.
Key words and phrases. von Neumann algebra, locally measurable operator, order topology, local convergence in measure.

natural topology that endows these $*$-algebras with the structure of a topological $*$-algebra is the measure topology induced by the trace τ [1]. If τ is a semifinite but not a finite trace, then one can consider the τ-locally measure topology $t_{\tau l}$ and the weak τ-locally measure $t_{w\tau l}$ [7]. However, in the case where $\mathcal{M}$ is not of finite type, the multiplication is not jointly continuous in the two variables with respect to these topologies. In this connection, it makes sense to use, for the $*$-algebra $LS(\mathcal{M})$, the locally measure topology $t(\mathcal{M})$, which was defined in [4] for any von Neumann algebras and which endows $LS(\mathcal{M})$ with the structure of a complete topological $*$-algebra [6, § 3.5].

The natural partial order on the selfadjoint part $LS_h(\mathcal{M}) = \{T \in LS(\mathcal{M}) : T^* = T\}$ permits to define, on $LS_h(\mathcal{M})$, an order convergence, (o)-convergence, and the generated by it (o)-topology $t_o(\mathcal{M})$. If $\mathcal{M}$ is a commutative von Neumann algebra, $t(\mathcal{M}) \leqslant t_o(\mathcal{M})$ and $t(\mathcal{M}) = t_o(\mathcal{M})$ on $LS_h(\mathcal{M})$ if and only if $\mathcal{M}$ is of σ-finite algebra [12, Ch. V, § 6]. For non-commutative von Neumann algebras, such relations between the topologies $t(\mathcal{M})$ and $t_o(\mathcal{M})$ do not hold in general. For example, if $\mathcal{M} = \mathcal{B}(\mathcal{H})$, then $LS(\mathcal{M}) = \mathcal{M}$ and the topology $t(\mathcal{M})$ coincides with the uniform topology that is strictly stronger than the (o)-topology on $\mathcal{B}_h(\mathcal{H})$ if $\dim(\mathcal{H}) = \infty$ [6, § 3.5].

In this paper, we study relations between the topology $t(\mathcal{M})$ and the topologies $t_{\tau l}$, $t_{w\tau l}$, and $t_o(\mathcal{M})$. We find that the topologies $t(\mathcal{M})$ and $t_{\tau l}$ (resp. $t(\mathcal{M})$ and $t_{w\tau l}$) coincide on $S(\mathcal{M}, \tau)$ if and only if $\mathcal{M}$ is finite, and $t(\mathcal{M}) = t_o(\mathcal{M})$ on $LS_h(\mathcal{M})$ holds if and only if $\mathcal{M}$ is a σ-finite and finite. Moreover, it turns out that the topology $t_{\tau l}$ (resp. $t_{w\tau l}$) coincides with the (o)-topology on $S_h(\mathcal{M}, \tau)$ only for finite traces. We give necessary and sufficient conditions for the topology $t(\mathcal{M})$ to be locally convex (resp., normable). We show that (o)-convergence of sequences in $LS_h(\mathcal{M})$ and convergence in the topology $t(\mathcal{M})$ coincide if and only if the algebra $\mathcal{M}$ is an atomic and finite algebra.

We use the von Neumann algebra terminology, notations and results from [9, 10], and those that concern the theory of measurable and locally measurable operators from [4, 6].

1. Preliminaries

Let $\mathcal{H}$ be a Hilbert space over the field $\mathbf{C}$ of complex numbers, $\mathcal{B}(\mathcal{H})$ be the $*$-algebra of all bounded linear operators on $\mathcal{H}$, I be the identity operator on $\mathcal{H}$, $\mathcal{M}$ be a von Neumann subalgebra of $\mathcal{B}(\mathcal{H})$, $\mathcal{P}(\mathcal{M}) = \{P \in \mathcal{M} : P^2 = P = P^*\}$ be the lattice of all projections in $\mathcal{M}$, and $\mathcal{P}_{fin}(\mathcal{M})$ be the sublattice of its finite projections. The center of a von Neumann algebra $\mathcal{M}$ will be denoted by $\mathcal{Z}(\mathcal{M})$.

A closed linear operator T affiliated with a von Neumann algebra $\mathcal{M}$ and having everywhere dense domain $\mathfrak{D}(T) \subset \mathcal{H}$ is called *measurable* if there

exists a sequence $\{P_n\}_{n=1}^{\infty} \subset \mathcal{P}(\mathcal{M})$ such that $P_n \uparrow I$, $P_n(\mathcal{H}) \subset \mathfrak{D}(T)$, and $P_n^{\perp} = I - P_n \in \mathcal{P}_{fin}(\mathcal{M})$, $n = 1, 2, \ldots$

A set $S(\mathcal{M})$ of all measurable operators is a $*$-algebra with identity I over the field $\mathbf{C}$ [2]. It is clear that $\mathcal{M}$ is a $*$-subalgebra of $S(\mathcal{M})$.

A closed linear operator T affiliated with $\mathcal{M}$ and having an everywhere dense domain $\mathfrak{D}(T) \subset \mathcal{H}$ is called *locally measurable* with respect to $\mathcal{M}$ if there is a sequence $\{Z_n\}_{n=1}^{\infty}$ of central projections in $\mathcal{M}$ such that $Z_n \uparrow I$ and $TZ_n \in S(\mathcal{M})$ for all $n = 1, 2, \ldots$

The set $LS(\mathcal{M})$ of all locally measurable operators with respect to $\mathcal{M}$ is a $*$-algebra with identity I over the field $\mathbf{C}$ with respect to the same algebraic operations as in $S(\mathcal{M})$ [4]. Here, $S(\mathcal{M})$ is a $*$-subalgebra of $LS(\mathcal{M})$. If $\mathcal{M}$ is finite, or if $\mathcal{M}$ is a factor, the algebras $S(\mathcal{M})$ and $LS(\mathcal{M})$ coincide.

For every $T \in S(\mathcal{Z}(\mathcal{M}))$ there exists a sequence $\{Z_n\}_{n=1}^{\infty} \subset \mathcal{P}(\mathcal{Z}(\mathcal{M}))$ such that $Z_n \uparrow I$ and $TZ_n \in \mathcal{M}$ for all $n = 1, 2, \ldots$ This means that $T \in LS(\mathcal{M})$. Hence, $S(\mathcal{Z}(\mathcal{M}))$ is a $*$-subalgebra of $LS(\mathcal{M})$, and $S(\mathcal{Z}(\mathcal{M}))$ coincides with the center of the $*$-algebra $LS(\mathcal{M})$.

For every subset $E \subset LS(\mathcal{M})$, the sets of all selfadjoint (resp., positive) operators in E will be denoted by E_h (resp., E_+). The partial order in $LS_h(\mathcal{M})$ defined by its cone $LS_+(\mathcal{M})$ is denoted by $\leqslant$. For a net $\{T_\alpha\}_{\alpha \in A} \subset LS_h(\mathcal{M})$, the notation $T_\alpha \uparrow T$ (resp., $T_\alpha \downarrow T$), where $T \in LS_h(\mathcal{M})$, means that $T_\alpha \leqslant T_\beta$ (resp., $T_\beta \leqslant T_\alpha$) for $\alpha \leqslant \beta$ and $T = \sup\limits_{\alpha \in A} T_\alpha$ (resp., $T = \inf\limits_{\alpha \in A} T_\alpha$).

We say that a net $\{T_\alpha\}_{\alpha \in A} \subset LS_h(\mathcal{M})$ (o)-*converges* to an operator $T \in LS_h(\mathcal{M})$, denoted by $T_\alpha \xrightarrow{(o)} T$, if there exist nets $\{S_\alpha\}_{\alpha \in A}$ and $\{R_\alpha\}_{\alpha \in A}$ in $LS_h(\mathcal{M})$ such that $S_\alpha \leqslant T_\alpha \leqslant R_\alpha$ for all $\alpha \in A$ and $S_\alpha \uparrow T$, $R_\alpha \downarrow T$.

The strongest topology on $LS_h(\mathcal{M})$ for which (o)-convergence implies its convergence in the topology is called order topology, or the (o)-topology, and is denoted by $t_o(\mathcal{M})$. If $\mathcal{M} = L_\infty(\Omega, \Sigma, \mu)$, $\mu(\Omega) < \infty$, the (o)-convergence of sequences in $LS_h(\mathcal{M})$ coincides with almost everywhere convergence , and convergence in the (o)-topology, $t_o(\mathcal{M})$, with measure convergence [11, Ch. III, § 9].

Let T be a closed operator with dense domain $\mathfrak{D}(T)$ in $\mathcal{H}$, $T = U|T|$ the polar decomposition of the operator T, where $|T| = (T^*T)^{\frac{1}{2}}$ and U is the partial isometry in $\mathcal{B}(\mathcal{H})$ such that U^*U is the right support of T. It is known that $T \in LS(\mathcal{M})$ if and only if $|T| \in LS(\mathcal{M})$ and $U \in \mathcal{M}$ [6, § 2.3]. If T is a self-adjoint operator affiliated with $\mathcal{M}$, then the spectral family of projections $\{E_\lambda(T)\}_{\lambda \in \mathbf{R}}$ for T belongs to $\mathcal{M}$ [6, § 2.1].

Let us now recall the definition of the locally measure topology. Let first $\mathcal{M}$ be a commutative von Neumann algebra. Then $\mathcal{M}$ is $*$-isomorphic to the $*$-algebra $L_\infty(\Omega, \Sigma, \mu)$ of all essentially bounded measurable complex-valued functions defined on a measure space (Ω, Σ, μ) with the measure μ satisfying the direct sum property (we identify functions that are equal almost everywhere). The direct sum property of a measure μ means that the Boolean algebra of all projections of the $*$-algebra $L_\infty(\Omega, \Sigma, \mu)$ is order

complete, and for any nonzero $P \in \mathcal{P}(\mathcal{M})$ there exists a nonzero projection $Q \leqslant P$ such that $\mu(Q) < \infty$.

Consider the $*$-algebra $LS(\mathcal{M}) = S(\mathcal{M}) = L_0(\Omega, \Sigma, \mu)$ of all measurable almost everywhere finite complex-valued functions defined on (Ω, Σ, μ) (functions that are equal almost everywhere are identified). On $L_0(\Omega, \Sigma, \mu)$, define a locally measure topology $t(\mathcal{M})$, that is, the linear Hausdorff topology, whose base of neighborhoods around zero is given by

$$\begin{aligned} W(B,\varepsilon,\delta) = \{f \in\ & L_0(\Omega,\, \Sigma,\, \mu)\colon \text{ there exists a set } E \in \Sigma \text{ such that} \\ & E \subseteq B,\ \mu(B \setminus E) \leqslant \delta,\ f\chi_E \in L_\infty(\Omega, \Sigma, \mu),\ \|f\chi_E\|_{L_\infty(\Omega,\Sigma,\mu)} \leqslant \varepsilon\}, \end{aligned}$$

where $\varepsilon,\ \delta > 0$, $B \in \Sigma$, $\mu(B) < \infty$, and $\chi(\omega) = \begin{cases} 1, & \omega \in E, \\ 0, & \omega \ \notin\ E. \end{cases}$

Convergence of a net $\{f_\alpha\}$ to f in the topology $t(\mathcal{M})$, denoted by $f_\alpha \stackrel{t(\mathcal{M})}{\longrightarrow} f$, means that $f_\alpha\chi_B \longrightarrow f\chi_B$ in measure μ for any $B \in \Sigma$ with $\mu(B) < \infty$. It is clear that the topology $t(\mathcal{M})$ does not change if the measure μ is replaced with an equivalent measure. Denote by $t_h(\mathcal{M})$ the topology on $LS_h(\mathcal{M})$ induced by the topology $t(\mathcal{M})$ on $LS(\mathcal{M})$.

Proposition 1. *If $\mathcal{M}$ is a commutative von Neumann algebra, then*

$$t_h(\mathcal{M}) \leqslant t_o(\mathcal{M}).$$

Proof. It sufficient to prove that any net $\{f_\alpha\}_{\alpha\in A} \subset LS_h(\mathcal{M})$, which (o)-converges to zero, also converges to zero with respect to the topology $t_h(\mathcal{M})$. Choose a net $\{g_\alpha\}_{\alpha\in A} \subset LS_h(\mathcal{M})$ such that $g_\alpha \downarrow 0$ and $-g_\alpha \leqslant f_\alpha \leqslant g_\alpha$ for all $\alpha \in A$.

Let $B \in \Sigma$ and $\mu(B) < \infty$ (we identify $\mathcal{M}$ with $L_\infty(\Omega, \Sigma, \mu)$). Then

$$-g_\alpha\chi_B \leqslant f_\alpha\chi_B \leqslant g_\alpha\chi_B,\ \alpha \in A,$$

and, since $g_\alpha\chi_B \downarrow 0$, we have $g_\alpha\chi_B \to 0$ in measure μ. Consequently, $f_\alpha\chi_B \to 0$ in measure μ and, hence, $f_\alpha \stackrel{t_h(\mathcal{M})}{\longrightarrow} 0$. □

Let now $\mathcal{M}$ be an arbitrary von Neumann algebra. Identify the center $\mathcal{Z}(\mathcal{M})$ with the $*$-algebra $L_\infty(\Omega, \Sigma, \mu)$, and $LS(\mathcal{Z}(\mathcal{M}))$ with the $*$-algebra $L_0(\Omega, \Sigma, \mu)$. Denote by $L_+(\Omega,\, \Sigma,\, m)$ the set of all measurable real-valued functions defined on (Ω, Σ, μ) and taking values in the extended half-line $[0, \infty]$ (functions that are equal almost everywhere are identifed). It was shown in [2] that there exists a mapping $\mathcal{D}\colon \mathcal{P}(\mathcal{M}) \to L_+(\Omega, \Sigma, \mu)$ that possesses the following properties:

(i) $\mathcal{D}(P) = 0$ if and only if $P = 0$;
(ii) $\mathcal{D}(P) \in L_0(\Omega, \Sigma, \mu) \Longleftrightarrow P \in \mathcal{P}_{fin}(\mathcal{M})$;
(iii) $\mathcal{D}(P \vee Q) = \mathcal{D}(P) + \mathcal{D}(Q)$ if $PQ = 0$;
(iv) $\mathcal{D}(U^*U) = \mathcal{D}(UU^*)$ for any partial isometry $U \in \mathcal{M}$;
(v) $\mathcal{D}(ZP) = Z\mathcal{D}(P)$ for any $Z \in \mathcal{P}(\mathcal{Z}(\mathcal{M}))$ and $P \in \mathcal{P}(\mathcal{M})$;

(vi) if $\{P_\alpha\}_{\alpha\in A}, P \in \mathcal{P}(\mathcal{M})$ and $P_\alpha \uparrow P$, then $\mathcal{D}(P) = \sup\limits_{\alpha\in A} \mathcal{D}(P_\alpha)$.

A mapping $\mathcal{D}\colon \mathcal{P}(\mathcal{M}) \to L_+(\Omega, \Sigma, \mu)$ that satisfies properties (i)—(vi) is called a *dimension function* on $\mathcal{P}(\mathcal{M})$.

For arbitrary numbers $\varepsilon, \delta > 0$ and a set $B \in \Sigma$, $\mu(B) < \infty$, set

$$V(B,\varepsilon,\delta) = \{T \in LS(\mathcal{M})\colon \text{ there exist } P \in \mathcal{P}(\mathcal{M}),\ Z \in \mathcal{P}(\mathcal{Z}(\mathcal{M})),$$
$$\text{such that } TP \in \mathcal{M}, \|TP\|_{\mathcal{M}} \leqslant \varepsilon,\ Z^\perp \in W(B,\varepsilon,\delta),\ \mathcal{D}(ZP^\perp) \leqslant \varepsilon Z\},$$

where $\|\cdot\|_{\mathcal{M}}$ is the C^*-norm on $\mathcal{M}$.

It was shown in [4] that the system of sets

$$\{\{T + V(B,\, \varepsilon,\, \delta)\}\colon\ T \in LS(\mathcal{M}),\ \varepsilon,\ \delta > 0,\ B \in \Sigma,\ \mu(B) < \infty\} \tag{1}$$

defines a linear Hausdorff topology $t(\mathcal{M})$ on $LS(\mathcal{M})$ such that sets (1) form a neighborhood base of the operator $T \in LS(\mathcal{M})$. Here, $(LS(\mathcal{M}), t(\mathcal{M}))$ is a complete topological $*$-algebra, and the topology $t(\mathcal{M})$ does not depend on a choice of the dimension function $\mathcal{D}$.

The topology $t(\mathcal{M})$ is called a *locally measure topology* [4].

We will need the following criterion for convergence of nets with respect to this topology.

Proposition 2 ([6, § 3.5]). (i) *A net $\{P_\alpha\}_{\alpha\in A} \subset \mathcal{P}(\mathcal{M})$ converges to zero with respect to the topology $t(\mathcal{M})$ if and only if there is a net $\{Z_\alpha\}_{\alpha\in A} \subset \mathcal{P}(\mathcal{Z}(\mathcal{M}))$ such that $Z_\alpha P_\alpha \in \mathcal{P}_{fin}(\mathcal{M})$ for all $\alpha \in A$, $Z_\alpha^\perp \xrightarrow{t(\mathcal{Z}(\mathcal{M}))} 0$, and $\mathcal{D}(Z_\alpha P_\alpha) \xrightarrow{t(\mathcal{Z}(\mathcal{M}))} 0$, where $t(\mathcal{Z}(\mathcal{M}))$ is the locally measure topology on $LS(\mathcal{Z}(\mathcal{M}))$.*

(ii) *A net $\{T_\alpha\}_{\alpha\in A} \subset LS(\mathcal{M})$ converges to zero with respect to the topology $t(\mathcal{M})$ if and only if $E_\lambda^\perp(|T_\alpha|) \xrightarrow{t(\mathcal{M})} 0$ for any $\lambda > 0$, where $\{E_\lambda^\perp(|T_\alpha|)\}$ is a spectral projection family for the operator $|T_\alpha|$.*

It follows from Proposition 2 that the topology $t(\mathcal{M})$ induces the topology $t(\mathcal{Z}(\mathcal{M}))$ on $LS(\mathcal{Z}(\mathcal{M}))$; hence, $S(\mathcal{Z}(\mathcal{M}))$ is a closed $*$-subalgebra of $(LS(\mathcal{M}), t(\mathcal{M}))$.

It is clear that

$$X \cdot V(B,\varepsilon,\delta) \subset V(B,\varepsilon,\delta)$$

for any $X \in \mathcal{M}$ with the norm $\|X\|_{\mathcal{M}} \leqslant 1$. Since $V^*(B,\varepsilon,\delta) \subset V(B,2\varepsilon,\delta)$ [6, § 3.5], we have

$$V(B,\varepsilon,\delta) \cdot Y \subset V(B,4\varepsilon,\delta)$$

for all $Y \in \mathcal{M}$ satisfying $\|Y\|_{\mathcal{M}} \leqslant 1$. Hence,

$$X \cdot V(B,\varepsilon,\delta) \cdot Y \subset V(B,4\varepsilon,\delta) \tag{2}$$

for any $\varepsilon, \delta > 0,\ B \in \Sigma$, $\mu(B) < \infty$, $X, Y \in \mathcal{M}$ with $\|X\|_{\mathcal{M}} \leqslant 1$, $\|Y\|_{\mathcal{M}} \leqslant 1$.

Since the involution is continuous in the topology $t(\mathcal{M})$, the set $LS_h(\mathcal{M})$ is closed in $(LS(\mathcal{M}), t(\mathcal{M}))$. The cone $LS_+(\mathcal{M})$ of positive elements is also closed in $(LS(\mathcal{M}), t(\mathcal{M}))$ [4]. Hence, for every increasing (or decreasing)

net $\{T_\alpha\}_{\alpha\in A} \subset LS_h(\mathcal{M})$ that converges to T in the topology $t(\mathcal{M})$, we have that $T \in LS_h(\mathcal{M})$ and $T = \sup_{\alpha\in A} T_\alpha$ (resp. $T = \inf_{\alpha\in A} T_\alpha$) [13, Ch. V, § 4].

2. Comparison of the topologies $t(\mathcal{M})$ and $t_o(\mathcal{M})$

Let $\mathcal{M}$ be an arbitrary von Neumann algebra, $t_o(\mathcal{M})$ be the (o)-topology on $LS_h(\mathcal{M})$. As before, $t_h(\mathcal{M})$ denotes the topology on $LS_h(\mathcal{M})$ induced by the topology $t(\mathcal{M})$ on $LS(\mathcal{M})$.

Theorem 1. *The following conditions are equivalent:*

(i) $t_h(\mathcal{M}) \leqslant t_o(\mathcal{M})$*;*

(ii) $\mathcal{M}$ *is finite.*

Proof. $(i) \Rightarrow (ii)$. Suppose that $\mathcal{M}$ is not finite. Then there is a sequence of pairwise orthogonal and pairwise equivalent projections $\{P_n\}_{n=1}^\infty$ in $\mathcal{P}(\mathcal{M})$. Choose a partial isometry U_n in $\mathcal{M}$ such that $U_n^*U_n = P_1,\ U_nU_n^* = P_n,\ n = 1, 2, \ldots$ Set $Q_n = \sup_{j\geqslant n} P_j$. Then $Q_n \in \mathcal{P}(\mathcal{M})$ and $Q_n \downarrow 0$. By condition (i) we have $Q_n \xrightarrow{t_h(\mathcal{M})} 0$. Since $P_n = P_nQ_n$, it follows from (2) that $P_n \xrightarrow{t_h(\mathcal{M})} 0$. Again using (2) we get that $P_1 = U_n^*P_nU_n \xrightarrow{t_h(\mathcal{M})} 0$ and, hence, $P_1 = 0$, which is not true. Consequently, $\mathcal{M}$ is finite.

$(ii) \Rightarrow (i)$. Let $\mathcal{M}$ be a finite von Neumann algebra, $\Phi : \mathcal{M} \mapsto \mathcal{Z}(\mathcal{M})$ a center-valued trace on $\mathcal{M}$ [10, Ch. V, § 2]. The restriction $\mathcal{D}$ of the trace Φ on $\mathcal{P}(\mathcal{M})$ is a dimension function on $\mathcal{P}(\mathcal{M})$. Let $\{T_\alpha\}_{\alpha\in A} \subset LS_h(\mathcal{M})$ and $T_\alpha \xrightarrow{(o)} 0$. Then there exists a net $\{S_\alpha\}_{\alpha\in A}$ in $LS_h(\mathcal{M})$ such that $S_\alpha \downarrow 0$ and $-S_\alpha \leqslant T_\alpha \leqslant S_\alpha$ for all $\alpha \in A$. Fix $\alpha_0 \in A$ and set $X_\alpha = XT_\alpha X,\ Y_\alpha = XS_\alpha X$ for $\alpha \geqslant \alpha_0$, where $X = (I + S_{\alpha_0})^{-\frac{1}{2}}$. It is clear that $-I \leqslant -Y_\alpha \leqslant X_\alpha \leqslant Y_\alpha \leqslant I$ for $\alpha \geqslant \alpha_0$ and $Y_\alpha \downarrow 0$. Consequently, $-I \leqslant -\Phi(Y_\alpha) \leqslant \Phi(X_\alpha) \leqslant \Phi(Y_\alpha) \leqslant I$ and $\Phi(Y_\alpha) \downarrow 0$.

Let $E_\lambda^\perp(Y_\alpha) = \{Y_\alpha > \lambda\}$ be a spectral projection for Y_α corresponding to the interval $(\lambda, +\infty),\ \lambda > 0$. Since

$$\mathcal{D}(E_\lambda^\perp(Y_\alpha)) \leqslant \frac{1}{\lambda}\Phi(Y_\alpha),$$

it follows that $\mathcal{D}(E_\lambda^\perp(Y_\alpha)) \xrightarrow{(o)} 0$ in $\mathcal{Z}(\mathcal{M})$. By Proposition 1, we have that

$$\mathcal{D}(E_\lambda^\perp(Y_\alpha)) \xrightarrow{t(\mathcal{Z}(\mathcal{M}))} 0$$

for all $\lambda > 0$. Hence, Proposition 2 gives that $Y_\alpha \xrightarrow{t(\mathcal{M})} 0$.

Set $Z_\alpha = X_\alpha + Y_\alpha$. Repeating the previous reasoning and using the inequality $0 \leqslant Z_\alpha \leqslant 2Y_\alpha$ we get that $Z_\alpha \xrightarrow{t(\mathcal{M})} 0$. Consequently, $X_\alpha = Z_\alpha - Y_\alpha \xrightarrow{t(\mathcal{M})} 0$ and, hence, $T_\alpha = X^{-1}X_\alpha X^{-1} \xrightarrow{t(\mathcal{M})} 0$. Thus, $t_h(\mathcal{M}) \leqslant t_o(\mathcal{M})$. □

Remark 1. *In the proof of the implication $(i) \Rightarrow (ii)$ of Theorem 1, it was shown that convergence to zero, in the topology $t(\mathcal{M})$, of any sequence of projections in $\mathcal{P}(\mathcal{M})$, which decreases to zero, implies that $\mathcal{M}$ is finite.*

Let us now find conditions that would imply that the topologies $t_h(\mathcal{M})$ and $t_o(\mathcal{M})$ coincide on $LS_h(\mathcal{M})$. Recall that a von Neumann algebra $\mathcal{M}$ is called σ-finite if any family of nonzero mutually orthogonal projections in $\mathcal{P}(\mathcal{M})$ is at most countable. It is known that the topology $t(\mathcal{M})$ on $LS(\mathcal{M})$ is metrizable if and only if the center $\mathcal{Z}(\mathcal{M})$ is σ-finite [4].

Proposition 3. *If $\mathcal{Z}(\mathcal{M})$ is σ-finite, then $t_o(\mathcal{M}) \leqslant t_h(\mathcal{M})$.*

Proof. Choose a neighborhood basis $\{V_k\}_{k=1}^{\infty}$ of zero in $(LS(\mathcal{M}), t(\mathcal{M}))$ such that
$$V_{k+1} + V_{k+1} \subset V_k$$
for all k.

Let $\{T_n\}_{n=1}^{\infty} \subset LS_h(\mathcal{M})$ and $T_n \xrightarrow{t(\mathcal{M})} 0$. Using relation (2) and the polar decomposition $T_n = U_n|T_n|$ we see that $|T_n| \xrightarrow{t(\mathcal{M})} 0$. Choose a subsequence $|T_{n_k}| \in V_k$ and set $S_k = \sum\limits_{i=1}^{k} |T_{n_i}|$. It is clear that $S_m - S_{k+1} \in V_k$ for $m > k$. Hence, there exists an operator $S \in LS_h(\mathcal{M})$ such that $S_k \xrightarrow{t(\mathcal{M})} S$. The sequence $R_k = S - \sum\limits_{i=1}^{k} |T_{n_i}|$ decreases and $R_k \xrightarrow{t(\mathcal{M})} 0$. Since the cone $LS_+(\mathcal{M})$ of positive elements is closed in $(LS(\mathcal{M}), t(\mathcal{M}))$, we have $R_k \downarrow 0$ and
$$-R_{k-1} \leqslant -|T_{n_k}| \leqslant T_{n_k} \leqslant |T_{n_k}| \leqslant R_{k-1}.$$
Consequently, $T_{n_k} \xrightarrow{(o)} 0$. Thus, for any sequence $\{T_n\}_{n=1}^{\infty} \subset LS_h(\mathcal{M})$, which converges to $T \in LS_h(\mathcal{M})$ in the topology $t(\mathcal{M})$, there exists a subsequence $T_{n_k} \xrightarrow{(o)} T$. This means that $t_o(\mathcal{M}) \leqslant t_h(\mathcal{M})$. □

We now describe a class of von Neumann algebras $\mathcal{M}$ for which the topologies $t_o(\mathcal{M})$ and $t_h(\mathcal{M})$ coincide.

Theorem 2. *The following conditions are equivalent:*

(i) *$\mathcal{M}$ is finite and σ-finite;*

(ii) *$t_o(\mathcal{M}) = t_h(\mathcal{M})$.*

Proof. The implication $(i) \Rightarrow (ii)$ follows from Theorem 1 and Proposition 3.

$(ii) \Rightarrow (i)$. If $t_o(\mathcal{M}) = t_h(\mathcal{M})$, then the von Neumann algebra $\mathcal{M}$ is finite by Theorem 1. Let us show that the center $\mathcal{Z}(\mathcal{M})$ is σ-finite.

Let $\{Z_j\}_{j\in\Delta}$ be a family of nonzero pairwise orthogonal projections in $\mathcal{P}(\mathcal{Z}(\mathcal{M}))$ satisfying $\sup\limits_{j\in\Delta} Z_j = I$ and $\mu(Z_j) < \infty$ (as before, we identify the commutative von Neumann algebra $\mathcal{Z}(\mathcal{M})$ with $L_\infty(\Omega, \Sigma, \mu)$ and $LS(\mathcal{Z}(\mathcal{M}))$ with $L_0(\Omega, \Sigma, \mu)$). Denote by E a $*$-subalgebra in $L_0(\Omega, \Sigma, \mu)$ of all functions $f \in L_0(\Omega, \Sigma, \mu)$ satisfying $fZ_j = \lambda_j Z_j$ for some $\lambda_j \in \mathbf{C}$, $j \in \Delta$. It is clear that E is $*$-isomorphic to the $*$-algebra $\mathbf{C}^{\Delta} = \{\{\lambda_j\}_{j\in\Delta} : \lambda_j \in \mathbf{C}\}$, and E_h is isomorphic to the algebra $\mathbf{R}^{\Delta} = \{\{r_j\}_{j\in\Delta} : r_j \in \mathbf{R}\}$. Denote by t the Tychonoff topology of coordinate convergence in $\mathbf{R}^{\Delta}$, and identify E_h with

$\mathbf{R}^\Delta$. If $f_\alpha \in E_h$, then $f_\alpha \xrightarrow{t(\mathcal{M})} 0$ if and only if $f_\alpha Z_j \xrightarrow{\mu} 0$ for all $j \in \Delta$. This means that the topology $t(\mathcal{M})$ induces the Tychonoff topology t on E_h.

Let us show that any subset $G \subset E_h$, upper bounded in $LS_h(\mathcal{M})$, is upper bounded in E_h, and the least upper bounds for G in E_h and in $S_h(\mathcal{M}) = LS_h(\mathcal{M})$ are the same.

For any operator $T \in S_+(\mathcal{M})$ there exists a maximal commutative $*$-subalgebra $\mathcal{A}$ of $S(\mathcal{M})$ containing $\mathcal{Z}(\mathcal{M})$ and T. Since $\mathcal{M}$ is a finite von Neumann algebra, $\mathcal{N} = \mathcal{A} \cap \mathcal{M}$ is also a finite von Neumann algebra, and $\mathcal{A} = S(\mathcal{N})$. We also have that $\mathcal{Z}(\mathcal{M}) \subset \mathcal{N}$. It is clear that $S_h(\mathcal{Z}(\mathcal{M}))$ is a regular sublattice of $S_h(\mathcal{N})$, that is, the least upper bounds and the least lower bounds of bounded subsets of $S_h(\mathcal{Z}(\mathcal{M}))$ calculated in $S_h(\mathcal{N})$ and in $S_h(\mathcal{Z}(\mathcal{M}))$ coincide.

Let $G \subset E_h$ and $S \leqslant T$ for all $S \in G$. Then there exists a least upper bound $\sup G$ in $S_h(\mathcal{N})$, which, since $S_h(\mathcal{Z}(\mathcal{M}))$ is regular, belongs to $S_h(\mathcal{Z}(\mathcal{M}))$. Since E_h is a regular sublattice in $S_h(\mathcal{Z}(\mathcal{M}))$, $\sup G \in E_h$. Consequently, any net $\{S_\alpha\} \subset E_h$ that (o)-converges to S in $S_h(\mathcal{M})$ will be (o)-convergent to S in E_h. This means that the (o)-topology $t_o(\mathcal{M})$ in $S_h(\mathcal{M})$ induces the (o)-topology $t_o(E_h)$ in E_h. Since $t_o(\mathcal{M}) = t_h(\mathcal{M})$, the Tychonoff topology t coincides with the (o)-topology in $\mathbf{R}^\Delta$. Consequently, the set Δ is at most countable [12, Ch. V, § 6], that is, $\mathcal{Z}(\mathcal{M})$ is a σ-finite von Neumann algebra. Since the von Neumann algebra $\mathcal{M}$ is finite, $\mathcal{M}$ is also a σ-finite algebra [2]. $\square$

Proposition 3 and Theorems 1 and 2 give the following.

Corollary 1. *(i) If $\mathcal{M}$ is a σ-finite von Neumann algebra but is not finite, then $t_o(\mathcal{M}) < t_h(\mathcal{M})$.*

(ii) If $\mathcal{M}$ is not a σ-finite von Neumann algebra but is finite, then $t_h(\mathcal{M}) < t_o(\mathcal{M})$.

Using Corollary 1 one can easily construct an example of a von Neumann algebra $\mathcal{M}$ for which the topologies $t_o(\mathcal{M})$ and $t_h(\mathcal{M})$ are incomparable.

Let $\mathcal{M}_1$ be a σ-finite von Neumann algebra which is not finite, $\mathcal{M}_2$ be a not σ-finite von Neumann algebra which is finite, and $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$. Then $LS(\mathcal{M}) = LS(\mathcal{M}_1) \times LS(\mathcal{M}_2)$ [6, § 2.5], and the topology $t(\mathcal{M})$ coincides with the product of the topologies $t(\mathcal{M}_1)$ and $t(\mathcal{M}_2)$. Moreover, a net $\{(T_\alpha^{(1)}, T_\alpha^{(2)})\}_{\alpha \in A}$ in $LS_h(\mathcal{M}_1) \times LS_h(\mathcal{M}_2)$ is (o)-convergent to an element $(T^{(1)}, T^{(2)}) \in LS_h(\mathcal{M}_1) \times LS_h(\mathcal{M}_2)$ if and only if the net $\{T_\alpha^{(k)}\}_{\alpha \in A}$ is (o)-convergent to $T^{(k)}$, $k = 1, 2$. Identifying $LS_h(\mathcal{M}_1)$ with the linear subspace $LS_h(\mathcal{M}_1) \times \{0\}$ and $LS_h(\mathcal{M}_2)$ with $\{0\} \times LS_h(\mathcal{M}_2)$ we get that the (o)-topology $t_o(\mathcal{M})$ in $LS_h(\mathcal{M})$ induces (o)-topologies in $LS_h(\mathcal{M}_1)$ and $LS_h(\mathcal{M}_2)$, correspondingly. It remains to apply Corollary 1, by which the topologies $t_o(\mathcal{M})$ and $t_h(\mathcal{M})$ are incomparable.

3. Locally measure topology on semifinite von Neumann algebras

Let $\mathcal{M}$ be a semifinite von Neumann algebra acting on a Hilbert space $\mathcal{H}$, τ be a faithful normal semifinite trace on $\mathcal{M}$. An operator $T \in S(\mathcal{M})$ with domain $\mathfrak{D}(T)$ is called *τ-measurable* if for any $\varepsilon > 0$ there exists a projection $P \in \mathcal{P}(\mathcal{M})$ such that $P(\mathcal{H}) \subset \mathfrak{D}(T)$ and $\tau(P^{\perp}) < \varepsilon$.

A set $S(\mathcal{M}, \tau)$ of all τ-measurable operators is a $*$-subalgebra of $S(\mathcal{M})$, and $\mathcal{M} \subset S(\mathcal{M}, \tau)$. If the trace τ is finite, then $S(\mathcal{M}, \tau) = S(\mathcal{M})$.

Let t_τ be *a measure topology* [1] on $S(\mathcal{M}, \tau)$ whose base of neighborhoods around zero is given by

$$U(\varepsilon, \delta) = \{T \in S(\mathcal{M}, \tau) : \quad \text{there exists a projection} \quad P \in \mathcal{P}(\mathcal{M}),$$
$$\text{such that } \tau(P^{\perp}) \leqslant \delta,\ TP \in \mathcal{M}, \quad \|TP\|_{\mathcal{M}} \leqslant \varepsilon\}, \quad \varepsilon > 0,\ \delta > 0.$$

The pair $(S(\mathcal{M}, \tau), t_\tau)$ is a complete metrizable topological $*$-algebra. Here, the topology t_τ majorizes the topology $t(\mathcal{M})$ on $S(\mathcal{M}, \tau)$ and, if τ is a finite trace, the topologies t_τ and $t(\mathcal{M})$ coincide [6, §§ 3.4, 3.5]. Denote by $t(\mathcal{M}, \tau)$ the topology on $S(\mathcal{M}, \tau)$ induced by the topology $t(\mathcal{M})$. It is not true in general that, if the topologies t_τ and $t(\mathcal{M}, \tau)$ are the same, then the von Neumann algebra $\mathcal{M}$ is finite. Indeed, if $\mathcal{M} = \mathcal{B}(\mathcal{H})$, $\dim(\mathcal{H}) = \infty$, $\tau = tr$ is the canonical trace on $\mathcal{B}(\mathcal{H})$, then $LS(\mathcal{M}) = S(\mathcal{M}) = S(\mathcal{M}, \tau) = \mathcal{M}$, and the two topologies t_τ and $t(\mathcal{M})$ coincide with the uniform topology on $\mathcal{B}(\mathcal{H})$.

At the same time, we have the following.

Proposition 4. *If $\mathcal{M}$ is a finite von Neumann algebra with a faithful normal semifinite trace τ and $t_\tau = t(\mathcal{M}, \tau)$, then $\tau(I) < \infty$.*

Proof. If $\tau(I) = \infty$, then there exists a sequence of projections $\{P_n\}_{n=1}^{\infty} \subset \mathcal{P}(\mathcal{M})$ such that $P_n \downarrow 0$ and $\tau(P_n) = \infty$. By Theorem 1, $P_n \xrightarrow{t(\mathcal{M})} 0$, however, $\{P_n\}_{n=1}^{\infty}$ does not converge to zero in the topology t_τ. □

Denote by $t_{h\tau}$ the topology on $S_h(\mathcal{M}, \tau)$ induced by the topology t_τ, and by $t_{o\tau}(\mathcal{M})$ the (o)-topology on $S_h(\mathcal{M}, \tau)$. The topology $t_{o\tau}(\mathcal{M})$, in general, does not coincide with the topology induced by the (o)-topology $t_o(\mathcal{M})$ on $S_h(\mathcal{M}, \tau)$. For example, for

$$\mathcal{M} = l_\infty(\mathbf{C}) = \{\{\alpha_n\}_{n=1}^{\infty} \subset \mathbf{C} : \sup_{n \geqslant 1} |\alpha_n| < \infty\}$$

and

$$\tau(\{\alpha_n\}) = \sum_{n=1}^{\infty} \alpha_n, \quad \alpha_n \geqslant 0,$$

we have that $LS(\mathcal{M}) = \mathbf{C}^{\mathbf{N}}$ and $S(\mathcal{M}, \tau) = l_\infty(\mathbf{C})$. Here, the (o)-topology $t_o(\mathcal{M})$ on $LS_h(\mathcal{M}) = \mathbf{R}^{\Delta}$ is the topology of the coordinatewise convergence, in particular,

$$T_n = n\, Z_n \xrightarrow{(o)} 0$$

in $\mathbf{R}^\Delta$, where $Z_n = \{0,\dots,0,1,0,\dots\}$, the number 1 is at the n-th place. However, the sequence $\{T_n\}_{n=1}^\infty$ does not converge in the (o)-topology $t_{o\tau}(l_\infty(\mathbf{C}))$, since any its subsequence is not bounded in $l_\infty(\mathbf{R}) = (l_\infty(\mathbf{C}))_h$ [11, Ch. VI, § 3].

Remark 2. *Since $S_+(\mathcal{M},\tau) = S(\mathcal{M},\tau) \cap LS_+(\mathcal{M})$ is closed in $(S(\mathcal{M},\tau), t_\tau)$ and $T_n \xrightarrow{t_\tau} T$ if and only if $|T_n - T| \xrightarrow{t_\tau} 0$* [6, § 3.4], *using metrizability of the topology t_τ and repeating the end of the proof of Proposition 3 we get that $t_{o\tau}(\mathcal{M}) \leqslant t_{h\tau}$.*

Remark 3. *Using the inclusions $U^*(\varepsilon,\delta) \subset U(\varepsilon, 2\delta)$ and $TU(\varepsilon,\delta) \subset U(\varepsilon\|T\|_{\mathcal{M}}, \delta)$, where $T \in \mathcal{M}$ we can see as in the proof of the implication $(i) \Rightarrow (ii)$ in Theorem 1 that the equality $t_{o\tau}(\mathcal{M}) = t_{h\tau}$ implies that the von Neumann algebra $\mathcal{M}$ is finite.*

Proposition 5. *Let $\mathcal{M}$ be a semifinite von Neumann algebra, τ be a faithful normal semifinite trace on $\mathcal{M}$. The following conditions are equivalent.*

- (i) *Any net that is (o)-convergent in $S_h(\mathcal{M},\tau)$ also converges in the topology $t_{h\tau}$;*
- (ii) $t_{o\tau}(\mathcal{M}) = t_{h\tau}$;
- (iii) $\tau(I) < \infty$.

Proof. The implication $(i) \Rightarrow (ii)$ follows from Remark 2 and definition of the topology $t_{o\tau}(\mathcal{M})$.

$(ii) \Rightarrow (iii)$. By Remark 3, the von Neumann algebra $\mathcal{M}$ is finite. Repeating the proof of Proposition 4 we see that $\tau(I) < \infty$.

$(iii) \Rightarrow (i)$. If $\tau(I) < \infty$, then $\mathcal{M}$ is a finite von Neumann algebra, and $S(\mathcal{M},\tau) = LS(\mathcal{M})$. Hence, (i) follows from Theorem 1. □

Together with the topology t_τ on $S(\mathcal{M},\tau)$, one can also consider two more Hausdorff vector topologies associated with the trace τ [7]. This are *the τ-locally measure topology $t_{\tau l}$* and *the weak τ-locally measure topology $t_{w\tau l}$*. The sets

$$U_\tau(\varepsilon,\delta,P) = \{T \in S(\mathcal{M},\tau) : \text{ there exists a projection } Q \in \mathcal{P}(\mathcal{M})$$
$$\text{such that } \; Q \leqslant P,\ \tau(P-Q) \leqslant \delta,\ TQ \in \mathcal{M},\ \ \|TQ\|_{\mathcal{M}} \leqslant \varepsilon\}$$

(resp.,

$$U_{w\tau}(\varepsilon,\delta,P) = \{T \in S(\mathcal{M},\tau) : \text{ there exists a projection } Q \in \mathcal{P}(\mathcal{M})$$
$$\text{such that } \; Q \leqslant P,\ \tau(P-Q) \leqslant \delta,\ QTQ \in \mathcal{M},\ \ \|QTQ\|_{\mathcal{M}} \leqslant \varepsilon\}),$$

where $\varepsilon > 0$, $\delta > 0$, $P \in \mathcal{P}(\mathcal{M})$, $\tau(P) < \infty$, form a neighborhood base around in the topology $t_{\tau l}$ (resp., in the topology $t_{w\tau l}$).

It is clear that $t_{w\tau l} \leqslant t_{\tau l} \leqslant t_\tau$, and if $\tau(I) < \infty$, all three topologies $t_{w\tau l}$, $t_{\tau l}$, and t_τ coincide.

Let us remark that if $\mathcal{M} = \mathcal{B}(\mathcal{H})$ and $\tau = tr$, the topology $t_{\tau l}$ coincides with the strong operator topology, and the topology $t_{w\tau l}$ with the weak

operator topology, that is, if $\dim(\mathcal{H}) = \infty$, we have $t_{w\tau l} < t_{\tau l} < t_\tau$ in this case.

The following criterion for convergence of nets in the topologies $t_{w\tau l}$ and $t_{\tau l}$ can be obtained directly from the definition.

Proposition 6 ([7]). *If $\{T_\alpha\}_{\alpha\in A}, T \subset S(\mathcal{M},\tau)$, then $T_\alpha \xrightarrow{t_{\tau l}} T$ (resp., $T_\alpha \xrightarrow{t_{w\tau l}} T$) if and only if $T_\alpha P \xrightarrow{t_\tau} TP$ (resp., $PT_\alpha P \xrightarrow{t_\tau} PTP$) for all $P \in \mathcal{P}(\mathcal{M})$ satisfying $\tau(P) < \infty$.*

Let us also list the following useful properties of the topologies $t_{\tau l}$ and $t_{w\tau l}$.

Proposition 7 ([7]). *Let T_α, $S_\alpha \in S(\mathcal{M},\tau)$. Then*

(i) $T_\alpha \xrightarrow{t_{\tau l}} \Longleftrightarrow |T_\alpha| \xrightarrow{t_{\tau l}} 0 \Longleftrightarrow |T_\alpha|^2 \xrightarrow{t_{w\tau l}} 0$;

(ii) if $T_\alpha \xrightarrow{t_{\tau l}} T$, $S_\alpha \xrightarrow{t_{\tau l}} S$, and the net $\{S_\alpha\}$ is t_τ-bounded, then $T_\alpha S_\alpha \xrightarrow{t_{\tau l}} TS$;

(iii) if $0 \leqslant S_\alpha \leqslant T_\alpha$ and $T_\alpha \xrightarrow{t_{w\tau l}} 0$, then $S_\alpha \xrightarrow{t_{w\tau l}} 0$.

To compare the topologies $t_{w\tau l}$ and $t_{\tau l}$ with the topology $t(\mathcal{M})$, we will need the following property of the topology $t(\mathcal{M})$.

Proposition 8. *The topology $t(\mathcal{M})$ induced the topology $t(P\mathcal{M}P)$ on $LS(P\mathcal{M}P)$, where $0 \neq P \in \mathcal{P}(\mathcal{M})$.*

Proof. Let $\{Q_\alpha\}_{\alpha\in A} \subset \mathcal{P}(P\mathcal{M}P)$ and $Q_\alpha \xrightarrow{t(\mathcal{M})} 0$. By Proposition 2$(i)$ there exists a net $\{Z_\alpha\}_{\alpha\in A} \subset \mathcal{P}(\mathcal{Z}(\mathcal{M}))$ such that $Z_\alpha Q_\alpha \in \mathcal{P}_{fin}(\mathcal{M})$ for any $\alpha \in A$, $Z_\alpha^\perp \xrightarrow{t(\mathcal{Z}(\mathcal{M}))} 0$, and $\mathcal{D}(Z_\alpha Q_\alpha) \xrightarrow{t(\mathcal{Z}(\mathcal{M}))} 0$. The projection $R_\alpha = PZ_\alpha$ is in the center $\mathcal{Z}(P\mathcal{M}P)$ of the von Neumann algebra $P\mathcal{M}P$, and $R_\alpha Q_\alpha = Z_\alpha Q_\alpha$ is a finite projection in $P\mathcal{M}P$. Denote by $Z(P)$ the central support of the projection P. The mapping $\psi : P\mathcal{Z}(\mathcal{M}) \to Z(P)\mathcal{Z}(\mathcal{M})$ given by

$$\psi(PZ) = Z(P)Z, \quad Z \in \mathcal{Z}(\mathcal{M}),$$

is a $*$-isomorphism from $P\mathcal{Z}(\mathcal{M})$ onto $Z(P)\mathcal{Z}(\mathcal{M})$. Since $\mathcal{Z}(P\mathcal{M}P) = P\mathcal{Z}(\mathcal{M})$ [9, Sect. 3.1.5], the $*$-algebras $\mathcal{Z}(P\mathcal{M}P)$ and $Z(P)\mathcal{Z}(\mathcal{M})$ are $*$-isomorphic.

It is clear that

$$\mathcal{D}_P(Q) := Z(P)\mathcal{D}(Q), \quad Q \in \mathcal{P}(P\mathcal{M}P),$$

is a dimension function on $\mathcal{P}(P\mathcal{M}P)$, where $\mathcal{D}$ is the initial dimension function on $\mathcal{P}(\mathcal{M})$. We have that

$$\mathcal{D}_P(R_\alpha Q_\alpha) = \mathcal{D}_P(Z_\alpha Q_\alpha) = Z(P)\mathcal{D}(Z_\alpha Q_\alpha) \xrightarrow{t(Z(P)\mathcal{Z}(\mathcal{M}))} 0.$$

Moreover,

$$P - R_\alpha = P(I - Z_\alpha) \overset{\psi}{\longmapsto} Z(P)Z_\alpha^\perp \xrightarrow{t(Z(P)\mathcal{Z}(\mathcal{M}))} 0.$$

Hence, by Proposition 2 (i) we get that $Q_\alpha \xrightarrow{t(P\mathcal{M}P)} 0$.

In the same way we can prove that $Q_\alpha \xrightarrow{t(P\mathcal{M}P)} 0$, $\{Q_\alpha\} \subset \mathcal{P}(P\mathcal{M}P)$, implies that $Q_\alpha \xrightarrow{t(\mathcal{M})} 0$.

Let now $\{T_\alpha\} \subset LS(P\mathcal{M}P)$ and $T_\alpha \xrightarrow{t(\mathcal{M})} 0$. By Proposition 2(ii), we have that $E_\lambda^\perp(|T_\alpha|) \xrightarrow{t(\mathcal{M})} 0$ for any $\lambda > 0$, where $\{E_\lambda(|T_\alpha|)\}$ is a family of spectral projections for $|T_\alpha|$. Denote by $\{E_\lambda^P(|T_\alpha|)\}$ the family of spectral projections for $|T_\alpha|$ in $LS(P\mathcal{M}P)$, $\lambda > 0$. It is clear that $E_\lambda(|T_\alpha|) = P^\perp + E_\lambda^P(|T_\alpha|)$ and $E_\lambda^\perp(|T_\alpha|) = P - E_\lambda^P(|T_\alpha|)$ for all $\lambda > 0$. It follows from above that $P - E_\lambda^P(|T_\alpha|) \xrightarrow{t(P\mathcal{M}P)} 0$ for all $\lambda > 0$. Hence, by Proposition 2(ii), it follows that $T_\alpha \xrightarrow{t(P\mathcal{M}P)} 0$.

One can similarly prove that the convergence $T_\alpha \xrightarrow{t(P\mathcal{M}P)} 0$ implies the convergence $T_\alpha \xrightarrow{t(\mathcal{M})} 0$. □

Theorem 3. *$t_{\tau l} \leqslant t(\mathcal{M},\tau)$.*

Proof. If $\{T_\alpha\} \subset S(\mathcal{M},\tau)$ and $T_\alpha \xrightarrow{t(\mathcal{M})} 0$, then $|T_\alpha|^2 \xrightarrow{t(\mathcal{M})} 0$. Let $P \in \mathcal{P}(\mathcal{M})$ and $\tau(P) < \infty$. By Proposition 8, we have that $P|T_\alpha|^2P \xrightarrow{t(P\mathcal{M}P)} 0$. Since $\tau(P) < \infty$, it follows that $LS(P\mathcal{M}P) = S(P\mathcal{M}P,\tau)$ and the topology $t(P\mathcal{M}P)$ coincides with the measure topology t_τ, that is, $P|T_\alpha|^2P \xrightarrow{t_\tau} 0$. By Proposition 6, we get that $|T_\alpha|^2 \xrightarrow{t_{w\tau l}} 0$. Hence, it follows from Proposition 7(i) that $T_\alpha \xrightarrow{t_{\tau l}} 0$. □

Remark 4. *It follows from Theorem 3 that the inequalities*

$$t_{w\tau l} \leqslant t_{\tau l} \leqslant t(\mathcal{M},\tau) \leqslant t_\tau$$

always hold. If $\mathcal{M} = \mathcal{B}(\mathcal{H}) \times L_\infty[0,\infty)$, $\tau((T,f)) = tr\,T + \int\limits_0^\infty f\,d\mu$, where $T \in \mathcal{B}_+(\mathcal{H})$, $0 \leqslant f \in L_\infty[0,\infty)$, μ is the linear Lebesgue measure on $[0,\infty)$, $\dim\mathcal{H} = \infty$, then $S(\mathcal{M},\tau) = \mathcal{B}(\mathcal{H}) \times S(L_\infty[0,\infty),\mu)$ and, in this case, the following strict inequalities hold:

$$t_{w\tau l} < t_{\tau l} < t(\mathcal{M},\tau) < t_\tau.$$

To find necessary and sufficient conditions for the topology $t(\mathcal{M},\tau)$ to coincide with the topologies $t_{\tau l}$ and $t_{w\tau l}$, we will need the following.

Proposition 9. *Let $\mathcal{M}$ be a semifinite von Neumann algebra, τ be a faithful normal semifinite trace on $\mathcal{M}$, $\{T_\alpha\}_{\alpha\in A} \subset \mathcal{M}$, $\sup\limits_{\alpha\in A}\|T_\alpha\|_\mathcal{M} \leqslant 1$.*

1) *If $\tau(I) < \infty$, then $T_\alpha \xrightarrow{t_\tau} 0$ if and only if $\tau(|T_\alpha|) \longrightarrow 0$.*
2) *If the algebra $\mathcal{M}$ if finite and $\Phi : \mathcal{M} \mapsto \mathcal{Z}(\mathcal{M})$ is a center-valued trace on $\mathcal{M}$, then the following conditions are equivalent:*
   (i) *$T_\alpha \xrightarrow{t_{\tau l}} 0$;*

(*ii*) $\Phi(|T_\alpha|) \stackrel{t(\mathcal{Z}(\mathcal{M}))}{\longrightarrow} 0$;

(*iii*) $T_\alpha \stackrel{t(\mathcal{M})}{\longrightarrow} 0$.

Proof. 1). If $\tau(|T_\alpha|) \to 0$, then it follows at once from the inequality $\tau(\{|T_\alpha| > \lambda\}) \leqslant \frac{1}{\lambda}\tau(|T_\alpha|)$, $\lambda > 0$, that $T_\alpha \stackrel{t_\tau}{\longrightarrow} 0$ (here we do not need the condition that $\sup\limits_{\alpha\in A} \|T_\alpha\|_{\mathcal{M}} \leqslant 1$). Conversely, let $T_\alpha \stackrel{t_\tau}{\longrightarrow} 0$. Then for every $\varepsilon > 0$ there exist $\alpha(\varepsilon) \in A$ and $P_\alpha \in \mathcal{P}(\mathcal{M})$, $\alpha \geqslant \alpha(\varepsilon)$, such that

$$\tau(P_\alpha^\perp) \leqslant \varepsilon, \quad T_\alpha P_\alpha \in \mathcal{M}, \quad \|T_\alpha P_\alpha\|_{\mathcal{M}} \leqslant \varepsilon.$$

Consequently, $\||T_\alpha|P_\alpha\|_{\mathcal{M}} \leqslant \varepsilon$ and $\tau(|T_\alpha|P_\alpha) \leqslant \varepsilon\tau(I)$. Whence,

$$\tau(|T_\alpha|) \leqslant \varepsilon\tau(I) + \tau(|T_\alpha|P_\alpha^\perp) \leqslant \varepsilon\tau(I) + \varepsilon \sup_{\alpha\in A} \|T_\alpha\|_{\mathcal{M}}$$

for all $\alpha \geqslant \alpha(\varepsilon)$, that is, $\tau(|T_\alpha|) \to 0$.

2). (*i*) $\Rightarrow$ (*ii*). If $T_\alpha \stackrel{t_{\tau l}}{\longrightarrow} 0$, then $|T_\alpha| \stackrel{t_{\tau l}}{\longrightarrow} 0$ (Proposition 7) and thus $|T_\alpha| \stackrel{t_{w\tau l}}{\longrightarrow} 0$. Let $\mathcal{P}_\tau(\mathcal{M}) = \{P \in \mathcal{P}(\mathcal{M}) : \tau(P) < \infty\}$. For every finite subset $\beta = \{P_1, P_2, \dots, P_n\} \subset \mathcal{P}_\tau(\mathcal{M})$, let $Q_\beta = \sup\limits_{1\leqslant i\leqslant n} P_i$. Denote by $B = \{\beta\}$ the directed set of all finite subsets of $\mathcal{P}_\tau(\mathcal{M})$, ordered by inclusion. It is clear that $Q_\beta \uparrow I$ and $Q_\beta \in \mathcal{P}_\tau(\mathcal{M})$ for all $\beta \in B$.

Let V, U be neighborhoods in $(S(\mathcal{Z}(\mathcal{M})), t(\mathcal{Z}(\mathcal{M})))$ of zero such that $V + V \subset U$ and $XV \subset V$ for any $X \in \mathcal{Z}(\mathcal{M})$ with $\|X\|_{\mathcal{M}} \leqslant 1$. Since $\Phi(Q_\beta^\perp) \downarrow 0$, there exists $\beta_0 \in B$ such that $\Phi(Q_{\beta_0}^\perp) \in V$. Since

$$0 \leqslant \Phi(Q_{\beta_0}^\perp|T_\alpha|Q_{\beta_0}^\perp) \leqslant \sup_{\alpha\in A} \|T_\alpha\|_{\mathcal{M}} \Phi(Q_{\beta_0}^\perp) \leqslant \Phi(Q_{\beta_0}^\perp),$$

we have that $\Phi(Q_{\beta_0}^\perp|T_\alpha|Q_{\beta_0}^\perp) \in V$ for all $\alpha \in A$. Identify the center $\mathcal{Z}(\mathcal{M})$ with $L_\infty(\Omega, \Sigma, \mu)$ and, for $E \in \Sigma$, $\mu(E) < \infty$, consider a faithful normal finite trace ν_E on $\chi_E\mathcal{M}$ defined by

$$\nu_E(X) = \int\limits_E \Phi(X) d\mu.$$

Since $|T_\alpha| \stackrel{t_{w\tau l}}{\longrightarrow} 0$, we have that $X_\alpha = Q_{\beta_0}|T_\alpha|Q_{\beta_0} \stackrel{t_\tau}{\longrightarrow} 0$. Consequently, $\chi_E X_\alpha \stackrel{t_\tau}{\longrightarrow} 0$ and, hence, $\chi_E X_\alpha \stackrel{t_{\nu_E}}{\longrightarrow} 0$. Using item 1) we see that

$$\int\limits_E \Phi(X_\alpha) d\mu = \nu(\chi_E X_\alpha) \longrightarrow 0.$$

Consequently, $\Phi(X_\alpha) \stackrel{t(\mathcal{Z}(\mathcal{M}))}{\longrightarrow} 0$. Hence, there exists $\alpha(V) \in A$ such that $\Phi(X_\alpha) \in V$ for all $\alpha \geqslant \alpha(V)$. Using that $\Phi(XY) = \Phi(YX)$ for $X, Y \in \mathcal{M}$ [9, Sect. 7.11] we get that

$$\Phi(|T_\alpha|) = \Phi(Q_{\beta_0}|T_\alpha|Q_{\beta_0}) + \Phi(Q_{\beta_0}^\perp|T_\alpha|Q_{\beta_0}^\perp) \in V + V \subset U$$

for $\alpha \geqslant \alpha(V)$, which implies the convergence $\Phi(|T_\alpha|) \stackrel{t(\mathcal{Z}(\mathcal{M}))}{\longrightarrow} 0$.

$(ii) \Rightarrow (iii)$. If $\Phi(|T_\alpha|) \xrightarrow{t(\mathcal{Z}(\mathcal{M}))} 0$, then it follows from $\Phi(E_\lambda^\perp(|T_\alpha|)) \leqslant \frac{1}{\lambda}\Phi(|T_\alpha|)$ that $\Phi(E_\lambda^\perp(|T_\alpha|)) \xrightarrow{t(\mathcal{Z}(\mathcal{M}))} 0$ for all $\lambda > 0$.

Setting $Z_\alpha = I$ and using Proposition 2(i) we get that $E_\lambda^\perp(|T_\alpha|) \xrightarrow{t(\mathcal{M})} 0$ for all $\lambda > 0$ and, hence, $T_\alpha \xrightarrow{t(\mathcal{M})} 0$, see Proposition 2(ii).

The implication $(iii) \Rightarrow (i)$ follows from Theorem 3. □

Theorem 4. *Let $\mathcal{M}$ be a semifinite von Neumann algebra, τ be a faithful normal semifinite trace on $\mathcal{M}$. The following condition are equivalent:*

(i) $t_{w\tau l} = t(\mathcal{M}, \tau)$;
(ii) $t_{\tau l} = t(\mathcal{M}, \tau)$;
(iii) $\mathcal{M}$ *is finite.*

Proof. $(i) \Rightarrow (ii)$. If $t_{w\tau l} = t(\mathcal{M}, \tau)$, then the operation of multiplication in $(S(\mathcal{M}, \tau), t_{w\tau l})$ is jointly continuous. In this case, as was shown in [7, Theorem 4.1], $t_{\tau l} = t_{w\tau l}$ and $\mathcal{M}$ is of finite type. The implication $(ii) \Rightarrow (iii)$ is proved similarly.

$(iii) \Rightarrow (i)$. Let $\mathcal{M}$ be a finite von Neumann algebra. Then $t_{w\tau l} = t_{\tau l}$ [7, Theorem 4.1]. Let $\{T_\alpha\} \subset S(\mathcal{M}, \tau)$ and $T_\alpha \xrightarrow{t_{w\tau l}} 0$. It follows from the identity $t_{w\tau l} = t_{\tau l}$ and Proposition 7 that $|T_\alpha| \xrightarrow{t_{w\tau l}} 0$.

If $\lambda \geqslant 1$, we have that $0 \leqslant E_\lambda^\perp(|T_\alpha|) \leqslant |T_\alpha|$, hence $E_\lambda^\perp(|T_\alpha|) \xrightarrow{t_{w\tau l}} 0$ by Proposition 7. Using Proposition 9, item 2, we get that $E_\lambda^\perp(|T_\alpha|) \xrightarrow{t(\mathcal{M})} 0$ for all $\lambda \geqslant 1$. Since

$$E_\lambda^\perp(|T_\alpha| E_1^\perp(|T_\alpha|)) = \begin{cases} E_1^\perp(|T_\alpha|), & 0 < \lambda < 1, \\ E_\lambda^\perp(|T_\alpha|), & \lambda \geqslant 1, \end{cases}$$

by Proposition $2(ii)$ we get that $|T_\alpha| E_1^\perp(|T_\alpha|) \xrightarrow{t(\mathcal{M})} 0$. Now, it follows from the inequality $|T_\alpha| E_1(|T_\alpha|) \leqslant |T_\alpha|$ that $|T_\alpha| E_1(|T_\alpha|) \xrightarrow{t_{w\tau l}} 0$. Since $t_{w\tau l} = t_{\tau l}$ and $\| \, |T_\alpha| E_1(|T_\alpha|) \, \|_{\mathcal{M}} \leqslant 1$, we have that $|T_\alpha| E_1(|T_\alpha|) \xrightarrow{t(\mathcal{M})} 0$ by Proposition 9, item 2.

Hence, $|T_\alpha| = |T_\alpha| E_1(|T_\alpha|) + |T_\alpha| E_1^\perp(|T_\alpha|) \xrightarrow{t(\mathcal{M})} 0$, and so $T_\alpha \xrightarrow{t(\mathcal{M})} 0$. With a use of Theorem 3 this shows that $t_{w\tau l} = t(\mathcal{M}, \tau)$. □

4. Comparison of the topologies $t_{\tau l}$ and $t_{w\tau l}$ with the (o)-topology on $S_h(\mathcal{M}, \tau)$

Let us denote by $t_{h\tau l}$ (resp., $t_{hw\tau l}$) the topology on $S_h(\mathcal{M}, \tau)$ induced by the topology $t_{\tau l}$ (resp., $t_{w\tau l}$), and find a connection between these topologies and the (o)-topology $t_{o\tau}(\mathcal{M})$.

Proposition 10. $t_{hw\tau l} \leqslant t_{h\tau l} \leqslant t_{o\tau}(\mathcal{M})$.

Proof. Let $\{T_\alpha\}_{\alpha \in A} \subset S_h(\mathcal{M}, \tau)$, $T_\alpha \downarrow 0$, $P \in \mathcal{P}(\mathcal{M})$, $\tau(P) < \infty$. Since $(P T_\alpha P) \downarrow 0$, we have that $P T_\alpha P \xrightarrow{t_{h\tau}} 0$ by Proposition 5. Consequently,

$T_\alpha \xrightarrow{t_{w\tau l}} 0$ by Proposition 6. Let $0 \leqslant S_\alpha \leqslant T_\alpha$, $S_\alpha \in S_h(\mathcal{M}, \tau)$. By Proposition 7(iii), $S_\alpha \xrightarrow{t_{w\tau l}} 0$ and, hence, $\sqrt{S_\alpha} \xrightarrow{t_{\tau l}} 0$ by Proposition 7(i). Let us show that $S_\alpha \xrightarrow{t_{\tau l}} 0$.

Let $\mu_t(T) = \inf\{\|TP\|_{\mathcal{M}} : P \in \mathcal{P}(\mathcal{M}),\ \tau(P^\perp) \leqslant t\}$, $t > 0$, be a non-increasing rearrangement of the operator T. Fix $\alpha_0 \in A$. For every $\alpha \geqslant \alpha_0$, we have

$$\mu_t(\sqrt{S_\alpha}) = \sqrt{\mu_t(S_\alpha)} \leqslant \sqrt{\mu_t(T_\alpha)} \leqslant \sqrt{\mu_t(T_{\alpha_0})},$$

in particular,

$$\sup_{\alpha \geqslant \alpha_0} \mu_t(\sqrt{S_\alpha}) \leqslant \sqrt{\mu_t(T_{\alpha_0})} < \infty$$

for all $t > 0$. Consequently, the net $\{\sqrt{S_\alpha}\}_{\alpha \geqslant \alpha_0}$ is t_τ-bounded [7, Lemma 1.2] and, hence, $S_\alpha \xrightarrow{t_{\tau l}} 0$ by Proposition 7(ii). Repeating now the proof of the implication $(ii) \Rightarrow (i)$ in Theorem 1 we get that $t_{h\tau l} \leqslant t_{o\tau}(\mathcal{M})$.

The inequality $t_{hw\tau l} \leqslant t_{h\tau l}$ follows from the inequality $t_{w\tau l} \leqslant t_{\tau l}$. $\square$

Corollary 2. *(i) If $t_{w\tau l} = t_\tau$ (resp., $t_{\tau l} = t_\tau$), then $\tau(I) < \infty$.*

(ii) If $t_{hw\tau l} = t_{h\tau l}$, then the algebra $\mathcal{M}$ is finite.

(iii) If $t_{hw\tau l} = t_{h\tau}$ (resp. $t_{h\tau l} = t_{h\tau}$), then $\tau(I) < \infty$.

Proof. (i). It follows from $t_{w\tau l} = t_\tau$ that $t_{\tau l} = t_\tau$. Consequently, $t_{h\tau} \leqslant t_{o\tau}(\mathcal{M})$ and, hence, $\tau(I) < \infty$ by Proposition 5.

(ii). If $T_\alpha \xrightarrow{t_{w\tau l}} 0$, then $T_\alpha^* \xrightarrow{t_{w\tau l}} 0$ and, hence,

$$Re\, T_\alpha = \frac{1}{2}(T_\alpha + T_\alpha^*) \xrightarrow{t_{w\tau l}} 0, \quad Im\, T_\alpha = \frac{1}{2i}(T_\alpha - T_\alpha^*) \xrightarrow{t_{w\tau l}} 0.$$

Since $t_{hw\tau l} = t_{h\tau l}$, we get that $T_\alpha \xrightarrow{t_{\tau l}} 0$. Hence, $t_{w\tau l} = t_{\tau l}$, which implies that $\mathcal{M}$ is finite [7].

Item (iii) follows from Propositions 5 and 10. $\square$

Corollary 3. *The following conditions are equivalent:*

(i) $t_{w\tau l} = t_\tau$;

(ii) $t_{\tau l} = t_\tau$;

(iii) $\tau(I) < \infty$.

Proof. The implication $(i) \Rightarrow (ii)$ follows from the inequalities $t_{w\tau l} \leqslant t_{\tau l} \leqslant t_\tau$, $(ii) \Rightarrow (iii)$ from Propositions 5 and 10, and the implication $(iii) \Rightarrow (i)$ is clear. $\square$

By Proposition 5, if $\tau(I) < \infty$, we have the following:

$$t_{hw\tau l} = t_{h\tau l} = t_{o\tau}(\mathcal{M}) = t_{h\tau}.$$

The following theorem permits to construct examples of von Neumann algebras $\mathcal{M}$ for which

$$t_{hw\tau l} < t_{h\tau l} < t_{o\tau}(\mathcal{M}) < t_{h\tau}.$$

Theorem 5. *If $t_{h\tau l} = t_{o\tau}(\mathcal{M})$, then $\tau(I) < \infty$.*

Proof. Assume that $\tau(I) = +\infty$ and first consider the σ-finite von Neumann algebra $\mathcal{M}$. In this case, there is a faithful normal positive linear functional φ on $\mathcal{M}$ [10, Ch. II, § 3]. If we have a net $\{T_\alpha\} \subset \mathcal{M}_+$ and $T_\alpha \downarrow 0$, then $\varphi(T_\alpha) \downarrow 0$ and, hence, there is a sequence of indices $\alpha_1 \leqslant \alpha_2 \leqslant \ldots$ such that $\varphi(T_{\alpha_n}) \downarrow 0$, which implies $T_{\alpha_n} \downarrow 0$.

Let a net $\{T_\alpha\} \subset S_h(\mathcal{M}, \tau)$ (o)-converge to zero in $S_h(\mathcal{M}, \tau)$. Choose $S_\alpha \in S_+(\mathcal{M}, \tau)$ such that $-S_\alpha \leqslant T_\alpha \leqslant S_\alpha$ and $S_\alpha \downarrow 0$. Fix α_0 and set $X = (I + S_{\alpha_0})^{-\frac{1}{2}}$, $Y_\alpha = X S_\alpha X$, $\alpha \geqslant \alpha_0$. Then $Y_\alpha \in \mathcal{M}_+$, $Y_\alpha \downarrow 0$, and hence there is a sequence $\alpha_1 \leqslant \alpha_2 \leqslant \ldots$ such that $Y_{\alpha_n} \downarrow 0$. Consequently, $S_{\alpha_n} \downarrow 0$ and $T_{\alpha_n} \xrightarrow{(o)} 0$.

Hence, the subset $F \subset S_h(\mathcal{M}, \tau)$ is closed in the (o)-topology $t_{o\tau}(\mathcal{M})$ if and only if F contains (o)-limits of all (o)-convergent sequences of elements in F.

Choose a sequence $\{P_n\}$ of nonzero pairwise orthogonal projections in $\mathcal{P}(\mathcal{M})$ satisfying $1 \leqslant \tau(P_n) < \infty$ and show that $F = \{\sqrt{n} P_n\}_{n=1}^\infty$ is closed in the (o)-topology $t_{o\tau}(\mathcal{M})$.

If $\{T_k\}_{k=1}^\infty \subset F$ is an (o)-convergent sequence of pairwise distinct elements, then $T_k = \sqrt{n_k} P_{n_k} \leqslant S$, $k = 1, 2, \ldots$, for some $S \in S_+(\mathcal{M}, \tau)$ and, hence, $0 \leqslant P_{n_k} \leqslant \frac{1}{\sqrt{n_k}} S \xrightarrow{t_\tau} 0$. Consequently, $\tau(P_{n_k}) \to 0$, which contradicts the inequality $\tau(P_{n_k}) \geqslant 1$, $k = 1, 2, \ldots$

Hence, the set F is closed in the (o)-topology $t_{o\tau}(\mathcal{M})$.

It remains to show that this set $F = \{\sqrt{n} P_n\}_{n=1}^\infty$ is not closed in the topology $t_{w\tau l}$ (here we do not use that the algebra $\mathcal{M}$ is σ-finite).

Denote by $\mathcal{M}_*^+$ the set of all positive normal linear functionals on $\mathcal{M}$, and let t_σ be the σ-strong topology on $\mathcal{M}$ generated by the family of seminorms $p_\psi(T) = \psi(T^*T)^{\frac{1}{2}}$, $\psi \in \mathcal{M}_*^+$, $T \in \mathcal{M}$ [10, Ch. II, § 2]. It is clear that the linear functional $\varphi_Q(T) = \tau(QTQ)$ belongs to $\mathcal{M}_*^+$ for all $Q \in \mathcal{P}_\tau(\mathcal{M})$ such that $\tau(Q) < \infty$. Thus, the convergence $T_\alpha \xrightarrow{t_\sigma} 0$ implies that $\tau(Q T_\alpha^* T_\alpha Q) = \varphi_Q(T_\alpha^* T_\alpha) = p_{\varphi_Q}^2(T_\alpha) \to 0$, that is, $|T_\alpha Q|^2 = Q T_\alpha^* T_\alpha Q \xrightarrow{t_\tau} 0$. By [14] we have that $|T_\alpha Q| \xrightarrow{t_\tau} 0$ and, hence, $T_\alpha Q \xrightarrow{t_\tau} 0$, that is, $T_\alpha \xrightarrow{t_{\tau l}} 0$ by Proposition 6. Consequently, the topology t_σ majorizes the topology $t_{\tau l}$ on $\mathcal{M}$.

Let us now show that $T = 0$ belongs to the closure of the set F in the topology t_σ. The sets

$$V(\varphi_1, \ldots, \varphi_n, \varepsilon) = \{T \in \mathcal{M} : p_{\varphi_i}(T) \leqslant \varepsilon,\ i = 1, 2, \ldots, n\}$$

form a neighborhood base around zero in the topology t_σ, where $\{\varphi_i\}_{i=1}^n \subset \mathcal{M}_*^+$, $\varepsilon > 0$, $n \in \mathbf{N}$. If $\varphi = \sum\limits_{i=1}^n \varphi_i$, then $\varphi \in \mathcal{M}_*^+$ and $p_{\varphi_i}(T) \leqslant p_\varphi(T)$, $i = 1, 2, \ldots, n$. Hence, the system of subsets $\{V(\varphi, \varepsilon) : \varphi \in \mathcal{M}_*^+, \varepsilon > 0\}$ is a neighborhood base around zero in the topology t_σ. If $V(\varphi, \varepsilon) \cap F = \emptyset$, then

$\varepsilon < p_\varphi(\sqrt{n}P_n) = \sqrt{n}\varphi(P_n)^{\frac{1}{2}}$ and, hence, $\varphi(P_n) > \frac{\varepsilon^2}{n}$ for all $n = 1, 2, \ldots$, which is impossible, since $\sum\limits_{n=1}^{\infty} \varphi(P_n) = \varphi(\sup\limits_{n\geqslant 1} P_n) < +\infty$.

Consequently, $V(\varphi,\varepsilon) \cap F \neq \emptyset$ for all $\varphi \in \mathcal{M}_*^+$, $\varepsilon > 0$. This means that $T = 0$ belongs to the closure of the set F in the topology t_σ. Since t_σ majorizes the topology $t_{\tau l}$ on $\mathcal{M}$, zero belongs to the closure of the set F in the topology $t_{\tau l}$. Consequently, the set F is not closed in $(S_h(\mathcal{M},\tau), t_{h\tau l})$ and, hence, $t_{h\tau l} < t_{o\tau}(\mathcal{M})$.

Let now $\mathcal{M}$ be a not σ-finite von Neumann algebra, $\{P_n\}_{n=1}^{\infty} \subset \mathcal{P}(\mathcal{M})$ be as before, $P = \sup\limits_{n\geqslant 1} P_n$, and $\mathcal{A} = P\mathcal{M}P$. It is clear that $\varphi(T) = \sum\limits_{n=1}^{\infty} \frac{\tau(P_nTP_n)}{2^n\tau(P_n)}$, $T \in \mathcal{A}$, is a faithful normal linear functional on $\mathcal{A}$, and thus the algebra $\mathcal{A}$ is σ-finite [10, Ch. II, § 3].

Let $\{T_\alpha\}_{\alpha\in A} \subset S_h(\mathcal{A},\tau)$, $T \in S_h(\mathcal{M},\tau)$, and $T_\alpha \xrightarrow{(o)} T$ in $S_h(\mathcal{M},\tau)$, that is, there is a net $\{S_\alpha\}_{\alpha\in A} \subset S_+(\mathcal{M},\tau)$ such that $-S_\alpha \leqslant T_\alpha - T \leqslant S_\alpha$ and $S_\alpha \downarrow 0$. Then

$$-PS_\alpha P \leqslant P(T_\alpha - T)P = T_\alpha - PTP \leqslant PS_\alpha P$$

and $PS_\alpha P \downarrow 0$, that is, $T_\alpha \xrightarrow{(o)} PTP$ in $S_h(\mathcal{A},\tau)$ and in $S_h(\mathcal{M},\tau)$. Consequently, $T = PTP$ so that $T \in S_h(\mathcal{A},\tau)$. This means that $S_h(\mathcal{A},\tau)$ is closed in $(S_h(\mathcal{M},\tau), t_{o\tau}(\mathcal{M}))$, and the (o)-topology $t_{o\tau}(\mathcal{M})$ induces the (o)-topology $t_{o\tau}(\mathcal{A})$ on $S_h(\mathcal{A},\tau)$. In particular, the set $F = \{\sqrt{n}P_n\}_{n=1}^{\infty}$ is closed in $(S_h(\mathcal{M},\tau), t_{o\tau}(\mathcal{M}))$, although it is not closed in the topology $t_{h\tau l}$. □

Proposition 10 and Theorem 5 immediately give the following.

Corollary 4. *(i) If $t_{hw\tau l} = t_{o\tau}(\mathcal{M})$, then $\tau(I) < \infty$.*
(ii) If $\mathcal{M}$ is not finite, then $t_{hw\tau l} < t_{h\tau l} < t_{o\tau}(\mathcal{M}) < t_{h\tau}$.
(iii) If $\mathcal{M}$ is finite and $\tau(I) = +\infty$, then $t_{hw\tau l} = t_{h\tau l} < t_{o\tau}(\mathcal{M}) < t_{h\tau}$.

5. The locally measure topology on atomic algebras

Necessary and sufficient conditions on the algebra $\mathcal{M}$ so that the topology $t_{\tau l}$ would be locally convex (resp., normable) were given in the paper of A. M. Bikchentaev [8]. Let us give a similar criterion for the topology $t(\mathcal{M})$.

A nonzero projection $P \in \mathcal{P}(\mathcal{M})$ is called an atom if $0 \neq Q \leqslant P$, $Q \in \mathcal{P}(\mathcal{M})$, implies that $Q = P$.

A von Neumann algebra $\mathcal{M}$ is atomic if every nonzero projection in $\mathcal{M}$ majorizes some atom. Any atomic von Neumann algebra $\mathcal{M}$ is $*$-isomorphic to the C^*-product

$$C^* - \prod_{j\in J} \mathcal{M}_j = \{\{T_j\}_{j\in J} : T_j \in \mathcal{M}_j,\ \sup_{j\in J} \|T_j\|_{\mathcal{M}_j} < +\infty\},$$

where $\mathcal{M}_j = \mathcal{B}(\mathcal{H}_j)$, $j \in J$. Since $LS(\mathcal{B}(\mathcal{H}_j)) = \mathcal{B}(\mathcal{H}_j)$ and $LS(C^* - \prod\limits_{j\in J} \mathcal{M}_j) = \prod\limits_{j\in J} LS(\mathcal{M}_j)$ [6, Ch. II, §3], we have that, for an atomic von

Neumann algebra $\mathcal{M}$, the $*$-algebra $LS(\mathcal{M})$ is $*$-isomorphic to the direct product $\prod\limits_{j\in J}\mathcal{B}(\mathcal{H}_j)$ of the algebras $\mathcal{B}(\mathcal{H}_j)$. By Proposition 2, the topology $t(\mathcal{M})$ coincides with the Tychonoff product of the topologies $t(\mathcal{B}(\mathcal{H}_j))$. Since $t(\mathcal{B}(\mathcal{H}_j))$ is a uniform topology $t_{\|\cdot\|_{\mathcal{B}(\mathcal{H}_j)}}$ on $\mathcal{B}(\mathcal{H}_j)$ generated by the norm $\|\cdot\|_{\mathcal{B}(\mathcal{H}_j)}$, the topology $t(\mathcal{M})$ is locally convex. For every $0 \leqslant \{T_j\}_{j\in J} \in C^* - \prod\limits_{j\in J}\mathcal{B}(\mathcal{H}_j)$, set $\tau(\{T_j\}_{j\in J}) = \sum\limits_{j\in J} tr_j(T_j)$, where tr_j is the canonical trace on $\mathcal{B}(\mathcal{H}_j)$. It is clear that τ is a faithful normal semifinite trace on the atomic von Neumann algebra $\mathcal{M} = C^* - \prod\limits_{j\in J}\mathcal{M}_j$, and the topology $t_{\tau l}$ is also locally convex [8], however, $t_{\tau l} \neq t(\mathcal{M})$ if $\dim\mathcal{H}_j = \infty$ for at least one index $j \in J$.

Proposition 11. *The topology $t(\mathcal{M})$ is locally convex if and only if $\mathcal{M}$ is $*$-isomorphic to the C^*-product $C^* - \prod\limits_{j\in J}\mathcal{M}_j$, where $\mathcal{M}_j$ are factors of type I or type III.*

Proof. Let $t(\mathcal{M})$ be a locally convex topology on $LS(\mathcal{M})$. Since $t(\mathcal{M})$ induces the topology $t(\mathcal{Z}(\mathcal{M}))$ on $\mathcal{Z}(\mathcal{M})$, we have that $(S(\mathcal{Z}(\mathcal{M})), t(\mathcal{Z}(\mathcal{M})))$ is a locally convex space. It follows from [12, Ch. V, §3] that $\mathcal{Z}(\mathcal{M})$ is an atomic von Neumann algebra. Hence, the algebra $\mathcal{M}$ is $*$-isomorphic to the C^*-product $C^* - \prod\limits_{j\in J}\mathcal{M}_j$, where $\mathcal{M}_j$ are factors for all $j \in J$. Let M_{j_0} be of type II-factor. Then there exists a nonzero finite projection $P \in \mathcal{P}(\mathcal{M})$ such that $P\mathcal{M}P$ is of type II_1. It follows from [12, Ch. V § 3] that $S(P\mathcal{M}P, t(P\mathcal{M}P))$ has not nonzero continuous linear functional and, hence, the topology $t(P\mathcal{M}P)$ can not be locally convex. By Proposition 8, the topology $t(\mathcal{M})$ can not be locally convex too. Consequently, $\mathcal{M}_j$ are either of type I or type III factors for all $j \in J$.

Conversely, let $\mathcal{M} = C^* - \prod\limits_{j\in J}\mathcal{M}_j$, where $\mathcal{M}_j$ are of type I or type III factors. Then $LS(\mathcal{M}_j) = \mathcal{M}_j$, $t(\mathcal{M}_j) = t_{\|\cdot\|_{\mathcal{M}_j}}$, $LS(\mathcal{M}) = \prod\limits_{j\in J}\mathcal{M}_j$ and, hence, the topology $t(\mathcal{M})$ is a Tychonoff product of the normed topologies $t(\mathcal{M}_j)$, that is, $t(\mathcal{M})$ is a locally convex topology. □

Corollary 5. *The topology $t(\mathcal{M})$ can be normed if and only if $\mathcal{M} = \prod\limits_{j=1}^{n}\mathcal{M}_j$, where $\mathcal{M}_j$ are of type I or type III factors, $j = 1, 2, ..., n$, and n is a positive integer.*

Proof. If the topology $t(\mathcal{M})$ is normable, then $(S(\mathcal{Z}(\mathcal{M})), t(\mathcal{Z}(\mathcal{M})))$ is a normable vector space. It follows from [12, Ch. V, § 3] that $\mathcal{Z}(\mathcal{M})$ is a finite dimensional algebra, which implies that $\mathcal{M} = \prod\limits_{j=1}^{n}\mathcal{M}_j$, where $\mathcal{M}_j$ are factors, $j = 1, 2, ..., n$. By Proposition 11, the factors $\mathcal{M}_j$ are either of type I or type III for all $j = 1, 2, ..., n$.

The converse implication is obvious. $\square$

Let us also mention one more useful property of the topologies $t(\mathcal{M})$ if $\mathcal{M}$ is an atomic finite algebra.

Proposition 12. *The following conditions are equivalent:*

(i) *$\mathcal{M}$ is an atomic finite von Neumann algebra;*

(ii) *if $\{T_n\}_{n=1}^{\infty} \subset LS_h(\mathcal{M})$, then $T_n \xrightarrow{t(\mathcal{M})} 0$ if and only if $T_n \xrightarrow{(o)} 0$.*

Proof. $(i) \Rightarrow (ii)$. Since $\mathcal{M}$ is a finite von Neumann algebra, it follows from $T_n \xrightarrow{(o)} 0$ that $T_n \xrightarrow{t(\mathcal{M})} 0$ by Theorem 1. Since $\mathcal{M}$ is atomic, we have $\mathcal{M} = C^* - \prod\limits_{j \in J} \mathcal{B}(\mathcal{H}_j)$. If $T_n = \{T_n^{(j)}\}_{j \in J}$, $T_n^{(j)} \in \mathcal{B}(H_j)$, and $T_n \xrightarrow{t(\mathcal{M})} 0$, then $\|T_n^{(j)}\|_{\mathcal{B}(\mathcal{H}_j)} \to 0$ as $n \to \infty$ for all $j \in J$. Since $|T_n^{(j)}| \leqslant \|T_n^{(j)}\|_{\mathcal{B}(\mathcal{H}_j)} \cdot I_{\mathcal{B}(\mathcal{H}_j)}$, it follows that $\{T_n^{(j)}\}_{n=1}^{\infty}$ (o)-converges to zero in $\mathcal{B}(\mathcal{H}_j)$ and, hence, $T_n \xrightarrow{(o)} 0$.

$(ii) \Rightarrow (i)$. It follows from Remark 1 that $\mathcal{M}$ is finite. Identify the center $\mathcal{Z}(\mathcal{M})$ with $L_\infty(\Omega, \Sigma, \mu)$. By condition (ii), any sequence in $L_0(\Omega, \Sigma, \mu)$ that μ-almost everywhere converges is convergent in the topology $t(\mathcal{M})$. Consequently, $\mathcal{Z}(\mathcal{M})$ is an atomic von Neumann algebra and, hence, $\mathcal{M} = C^* - \prod\limits_{j \in J} \mathcal{M}_j$, where $\mathcal{M}_j$ are finite factors of types I or II. If there is an index $j_0 \in J$ for which $\mathcal{M}_{j_0}$ is of type II, then there exists a nonzero projection $P \in \mathcal{P}(\mathcal{M})$, $\tau(P) = 1$, such that $P\mathcal{M}P$ is of type II_1.

Let $\mathcal{A}$ be a maximal commutative $*$-subalgebra of $P\mathcal{M}P$. Then $\mathcal{A}$ has no atoms and there is a collection $Q_n^{(k)}$, $1 \leqslant k \leqslant n$, of pairwise orthogonal projections in $\mathcal{P}(\mathcal{A})$ such that $\sup\limits_{1 \leqslant k \leqslant n} Q_n^{(k)} = P$ and $\tau(Q_n^{(k)}) = \frac{1}{n}$, $k = 1, 2, \ldots, n$. Set $X_n^{(k)} = nQ_n^{(k)}$, $k = 1, 2, \ldots, n$, and index the operators $X_n^{(k)}$ by setting $T_1 = X_1^{(1)}$, $T_2 = X_2^{(1)}$, $T_3 = X_2^{(2)}, \ldots$ It is clear that $T_n \xrightarrow{t_\tau} 0$ and, by condition (ii), $T_n \xrightarrow{(o)} 0$ in $LS_h(P\mathcal{M}P)$, that is, there exists a sequence $\{S_n\}_{n=1}^{\infty} \subset LS_+(P\mathcal{M}P)$ such that $S_n \downarrow 0$ and $0 \leqslant T_n \leqslant S_n$, $n = 1, 2, \ldots$

Since $E_n^{\perp}(S_1) \in P\mathcal{M}P$ and $E_n^{\perp}(S_1) \downarrow 0$, there is an index n_0 such that $\tau(E_{n_0}^{\perp}(S_1)) < \frac{1}{2}$. Set $E = PE_{n_0}(S_1)$ and $L_n = ES_nE$. It is clear that $\frac{1}{2} < \tau(E) \leqslant 1$, $0 \leqslant L_n \leqslant n_0 E$, $L_n \downarrow 0$, and $0 \leqslant ET_nE \leqslant L_n$, $n = 1, 2, \ldots$

Since $\tau(L_n) \downarrow 0$, the inequality $\tau(E_\varepsilon^{\perp}(L_n)) \leqslant \frac{1}{\varepsilon}\tau(L_n)$ implies that $\tau(E_\varepsilon^{\perp}(L_n)) \to 0$ for all $\varepsilon > 0$.

Fix $\varepsilon \in (0, 1)$ and choose an index n_1 such that $\tau(E_\varepsilon^{\perp}(L_{n_1})) < \frac{1}{2}$. For the projection $G = PE_\varepsilon(L_{n_1})$, we have that $GET_nEG \leqslant GL_nG \leqslant GL_{n_1}G \leqslant \varepsilon G$ for all $n \geqslant n_1$, that is, $\|Q_n^{(k)}EG\|_{\mathcal{M}} \leqslant \sqrt{\frac{\varepsilon}{n}}$ for any $n \geqslant n_1$, $k = 1, 2, \ldots, n$. If $E \wedge G = 0$, then $1 = \tau(P) \geqslant \tau(E \vee G) = \tau(E) + \tau(G) > 1$. Consequently, $E \wedge G \neq 0$, so that there exists a vector $\xi \in (E \wedge G)(\mathcal{H}) \subset P(\mathcal{H})$ with $\|\xi\|_{\mathcal{H}} = 1$, where $\|\cdot\|_{\mathcal{H}}$ is the norm on the Hilbert space $\mathcal{H}$ on which the von

Neumann algebra $\mathcal{M}$ acts. For each $n \geqslant n_1$, we have that

$$1 = \|P\xi\|^2_{\mathcal{H}} = \sum_{k=1}^{n} \|Q_n^{(k)}\xi\|^2_{\mathcal{H}} = \sum_{k=1}^{n} \|(Q_n^{(k)}EG)\xi\|^2_{\mathcal{H}} \leqslant \frac{\varepsilon}{n} + \frac{\varepsilon}{n} + \ldots + \frac{\varepsilon}{n} = \varepsilon < 1.$$

This contradiction shows that the sequence $\{T_n\}_{n=1}^{\infty}$ can not be (o)-convergent to zero in $LS_h(P\mathcal{M}P)$. Hence, all the factors $\mathcal{M}_j$, $j \in J$, are of type I, that is, $\mathcal{M}$ is an atomic von Neumann algebra. □

Corollary 6. *The following conditions are equivalent.*

(i) Any $t_{w\tau l}$-convergent sequence in $S_h(\mathcal{M},\tau)$ is (o)-convergent.
(ii) Any $t_{\tau l}$-convergent sequence in $S_h(\mathcal{M},\tau)$ is (o)-convergent.
(iii) Any t_{τ}-convergent sequence in $S_h(\mathcal{M},\tau)$ is (o)-convergent.
(iv) $\mathcal{M}$ is an atomic von Neumann algebra and $\tau(I) < \infty$.

Proof. The implications $(i) \Rightarrow (ii) \Rightarrow (iii)$ follow from the inequalities $t_{w\tau l} \leqslant t_{\tau l} \leqslant t_{\tau}$.

$(iii) \Rightarrow (iv)$. Since the topology t_τ is metrizable, it follows from Remark 2 that $t_{h\tau} = t_{o\tau}$ and, hence, $\tau(I) < \infty$ by Proposition 5, in particular, $t_\tau = t(\mathcal{M})$. It remains to apply Proposition 12.

The implication $(iv) \Rightarrow (i)$ follows from Theorem 4 and Proposition 12. □

Remark 5. *It was shown in the proof of the implication $(ii) \Rightarrow (i)$ in Proposition 12 that for a non-atomic von Neumann algebra $\mathcal{M}$ with a faithful normal trace τ there always exists a sequence $\{E_n\}_{n=1}^{\infty}$ of pairwise commuting projections in $\mathcal{P}(\mathcal{M})$ such that $E_n \xrightarrow{t_\tau} 0$, however, $\{E_n\}_{n=1}^{\infty}$ does not (o)-converge in $S_h(\mathcal{M},\tau)$.*

References

[1] E. Nelson. Notes on noncommutative integration. *J. Funct. Anal*, no. 15 (1974), 103–116.
[2] I. E. Segal. A non-commutative extension of abstract integration. *Ann. Math.* no. 57 (1953), 401–457.
[3] P .G. Dixon. Unbounded operator algebras. *Proc. London Math. Soc*, vol. 23, no. 3 (1973), 53–59.
[4] F. J. Yeadon. Convergence of measurable operators. *Proc. Camb. Phil. Soc*, no 74 (1974), 257–268.
[5] B. S. Zakirov, V. I. Chilin. Abstract characterization of EW^*-algebras. *Funkts. Anal. Prilozh*, vol. 25, no. 1 (1991), 76–78. (Russian).
[6] M. A. Muratov, I. I. Chilin. Algebras of measurable and locally measurable operators. Proc. Inst. Math. of Nat. Acad. Sci. of Ukraine, Kiev, vol. 69, 2007. (Russian).
[7] A. M. Bikchentaev. Local Convergence in Measure on Semifinite von Neumann Algebras. *Tr. Mat. Inst. Steklova*, vol.255 (2006), 41–54. (Russian).
[8] A. M. Bikchentaev. Local convergence in measure on semifinite von Neumann algebras, II. *Mat. Zametki*, vol. 82, no.5 (2007), 783–786. (Russian).
[9] S. Stratila, L. Zsido. Lectures on von Neumann algebras. Abacus Press, 1975.
[10] M. Takesaki. Theory of operator algebras I. Springer, New York, 1979.

[11] B. Z. Vulikh. Introduction to the theory of partially ordered spaces. Gordon and Breach, New York, 1967.

[12] T. A. Sarymsakov, Sh. A. Ayupov, D. Khadzhyev, V. I. Chilin. Ordered algebras. FAN, Tashkent, 1983. (Russian).

[13] H. Shaefer. Topological vector spaces. The Macmilla Company. New York Collier-Macmillan Limited, London, 1966.

[14] O. E. Tikhonov. Continuity of operator-valued functions in topologies connected with a trace on a von Neumann algebra. *Izv. Vyssh. Uchebn. Zav.* no. 1 (1987), 77–79. (Russian).

Vladimir Chilin
National University of Uzbekistan,
Tashkent, 100174, Republica of Uzbekistan.
E-mail: chilin@ucd.uz

Mustafa Muratov
Taurida National University,
Simferopol, 95007, Ukraine.
E-mail: mustafa-muratov@mail.ru